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1979 J. Phys. A: Math. Gen. 12 489

(http://iopscience.iop.org/0305-4470/12/4/009)

J. Phys. A: Math. Gen., Vol. 12, No. 4, 1979. Printed in Great Britain

Dynamical symmetries in a spherical geometry II

Howard I Leemon†

Department of Physics, University of Edinburgh, James Clerk Maxwell Building, The King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, U.K.

Received 13 September 1978

Abstract. The quantum mechanical Coulomb and isotropic oscillator problems in an N-dimensional spherical geometry, which were shown in the previous paper to possess the dynamical symmetry groups $SO(N+1)$ and $SU(N)$ respectively as classical systems, are analysed by the method used by Pauli to find the energy eigenvalues of the hydrogen atom. This analysis is carried through completely for $N=3$ to obtain energy eigenvalues and recurrence relations among energy eigenfunctions. It is shown that Pauli's method is equivalent to Schrödinger's method of solving the radial Schrödinger equation by factorisation of the second order differential operator. The latter method is used to find the energy eigenvalues in N dimensions, and the corresponding eigenfunctions are obtained in closed form.

1. Introduction

The Coulomb and isotropic oscillator problems in an N-dimensional spherical geometry have been considered in a previous paper (Higgs 1978). There it is shown that the classical systems possess the dynamical symmetry groups SO $(N+1)$ and $SU(N)$ respectively. The first of these groups is generated by the angular momentum L_{ij} and the generalisation of the Runge–Lenz vector R_i, while the second is generated by L_{ij} and the generalisation of the Fradkin tensor N_{ij}. These results are demonstrated explicitly by rewriting the classical Poisson bracket algebras in the required forms. This has not been done for the commutator algebras of the quantum mechanical operators (except in two dimensions) so that the group theoretical techniques used in the familiar flat-space problems are not applicable.

In this paper an alternative technique, due to Pauli (1926), is shown to provide the means of solution of the eigenvalue problems for these systems. In each case the algebra of the generators is replaced by the algebra of the matrix elements of the generators in a basis of energy and angular momentum eigenstates. The values of the matrix elements are obtained and from these the energy eigenvalues are found. Further, the technique can be extended to provide recurrence relations which generate the energy eigenfunctions completely. In § 2 the energy eigenvalues will be found for $N=3$. In § 3 the recurrence relations for the eigenfunctions will be found. In § 4 it is shown that this method is equivalent to a method of Schrödinger (1940) for solving some types of differential equations and the latter is used to find the energy eigenvalues and the recurrence relations in N dimensions. In § 5 these recurrence relations are used

† Present address: Department of Mathematics, University of Queensland, St. Lucia, Queensland 4067, Australia.

to generate the eigenfunctions in closed form. In § 6 it is shown that the energy eigenstates form bases for representations of the two symmetry groups although an explicit construction has not been achieved for either group.

2. The energy eigenvalues for $N=3$

When working with angular momentum eigenstates it is the spherical components of an operator rather than its Cartesian components which are most appropriate. These can be defined from the Cartesian components by

$$R_0 = R_3 \tag{1}$$

and

$$R_{\pm 1} = \mp 2^{-1/2}(R_1 \pm \mathrm{i} R_2) \tag{2}$$

for any vector R_i (and similarly for any pseudo-vector L_{ij}) and by

$$N_0 = (3^{1/2}/2) N_{33}, \tag{3}$$

$$N_{\pm 1} = \mp 2^{-1/2}(N_{13} \pm \mathrm{i} N_{23}) \tag{4}$$

and

$$N_{\pm 2} = 8^{-1/2}(N_{11} - N_{22} \pm 2\mathrm{i} N_{12}) \tag{5}$$

for any symmetric traceless tensor of rank 2, N_{ij}.

The basis states for matrix elements of these operators, denoted $|E, l, m\rangle$, are the familiar eigenstates of H, L^2 and L_0. These matrix elements can be separated by means of the Wigner–Eckart theorem (e.g. Edmonds 1957) into a coefficient containing the m dependence and a reduced matrix element involving only E and l. It is the algebra of these reduced matrix elements which we will be able to solve.

The equations which are needed are obtained from the commutation relations given by Higgs (1979 equations (17*a*), (33*a*) and (37)) and from the orthogonality conditions

$$\epsilon_{ijk} L_{ij} R_k = 0 \tag{6}$$

and

$$\epsilon_{jkl} N_{ij} L_{kl} = -(2/3\omega)(2H - \lambda L^2 + \tfrac{3}{2}\lambda) L_i .\dagger \tag{7}$$

The equations can all be written in spherical components but the reduction of the matrix elements allows us to choose only four. These are

$$[R_{-1}, R_{+1}] = (-2H + 2\lambda L^2) L_0, \tag{8}$$

$$-L_{+1} R_{-1} - L_{-1} R_{+1} + L_0 R_0 = 0, \tag{9}$$

$$\begin{aligned}[N_{-1}, N_{+1}] = \{1 - \tfrac{1}{4}(\lambda^2/\omega^2) + (2H - \lambda L^2)(\lambda/3\omega^2)\} L_0 \\ + (\lambda/2\omega)\{L_{+1} N_{-1} + L_{-1} N_{+1} + (2/3^{1/2}) L_0 N_0 \\ + N_{-1} L_{+1} + N_{+1} L_{-1} + (2/3^{1/2}) N_0 L_0)\end{aligned} \tag{10}$$

and

$$N_{-1} L_{+1} + N_{+1} L_{-1} - (2/3^{1/2}) N_0 L_0 = 3\omega^{-1}(2H - \lambda L^2 + \tfrac{3}{2}\lambda) L_0 . \tag{11}$$

† These equations correspond to the classical result that the angular momentum vector is normal to the plane containing the orbit and can be obtained by direct calculation from the definitions.

Now consider these two pairs of equations separately. Equation (9) provides the result†

$$E, l^{l'\dots l'}\|R\|E, l) = 0 \quad \text{unless } l^{l'\dots l'} = l \pm 1 \tag{12}$$

while equation (8) leads to

$$|(E, l\|R\|E, l+1)|^2/(l+1) - |(E, l-1\|R\|E, l)|^2/l = (2l+1)[2E - 2\lambda l(l+1)]. \tag{13}$$

The latter is a difference equation which has a general solution

$$|(E, l\|R\|E, l+1)|^2/(l+1) = C + 2E(l+1)^2 - \tau l(l+2)(l+1)^2. \tag{14}$$

The constant C is found to be μ^2 by evaluating the matrix elements of R^2 which has been found by Higgs (1978 equation (18*a*)).

As the value of l increases the right-hand side of (14) becomes negative while the left-hand side is non-negative. A state of higher l can always be generated from any state $|E, l, m\rangle$ by means of

$$\begin{aligned} R_\sigma|E, l, m\rangle = \langle E, l+1, m+\sigma|R_\sigma|E, l, m\rangle|E, l+1, m+\sigma\rangle \\ + \langle E, l-1, m+\sigma|R_\sigma|E, l, m\rangle|E, l-1, m+\sigma\rangle \end{aligned} \tag{15}$$

unless there exists an integer n such that

$$(E, n\|R\|E, n+1) = 0. \tag{16}$$

This leads immediately to the result that the energy has the discrete values

$$E_n = -\frac{1}{2}\frac{\mu^2}{(n+1)^2} + \frac{1}{2}\lambda n(n+2). \tag{17}$$

Conversely, for every non-negative integer n, the substitution of (17) into (14) shows that n is the maximum value of a sequence of l values. Thus all the energy eigenvalues have been found for the Coulomb potential.

Equations (10) and (11) can be used in a similar manner for the oscillator potential. Equation (11) gives us

$$(E, l^{l'\dots l'}\|N\|E, l) = 0 \quad \text{unless } l^{l'\dots l'} = l, l \pm 2 \tag{18}$$

and

$$(E, l\|N\|E, l) = (-1/3^{1/2}\omega)\left(\frac{(2l+2)(2l+1)(2l)}{(2l+3)(2l-1)}\right)^{1/2}\{E - \tfrac{1}{2}\lambda l(l+1) + \tfrac{3}{4}\lambda\} \tag{19}$$

while equation (10) leads to the result

$$\begin{aligned} &\frac{8|(E, l\|N\|E, l+2)|^2}{(2l+4)(2l+3)(2l+2)} \\ &\qquad = \frac{12(2l-1)}{(2l+3)}\frac{(E, l\|N\|E, l)^2}{(2l+2)(2l+1)(2l)} - \left(1 - \frac{\lambda^2}{4\omega^2} + \frac{2\lambda}{3\omega^2}[E - \tfrac{1}{2}\lambda l(l+1)]\right) \\ &\qquad\quad - \frac{2\lambda}{3^{1/2}\omega}\frac{(2l-1)(2l-3)(E, l\|N\|E, l)}{[(2l+3)(2l+2)(2l+1)(2l)(2l-1)]^{1/2}} \qquad (20) \\ &\qquad = \frac{[E - \frac{1}{2}\lambda(l^2 + 3l + \frac{3}{2})]^2 - (\omega^2 + \frac{1}{4}\lambda^2)(l + \frac{3}{2})^2}{\omega^2(l + \frac{3}{2})^2} \qquad (21) \end{aligned}$$

† The following equations are obtained using the definitions of Edmonds (1957).

on the substitution of (19). The right-hand side of (21) can be rewritten as

$$\frac{\lambda^2}{\omega^2(2l+3)^2}\left\{\left[\left(l+\frac{3}{2}\right)^2-\left(\frac{3}{4}+\frac{2E}{\lambda}\right)\right]^2-\frac{4(\omega^2+\frac{1}{4}\lambda^2)(l+\frac{3}{2})^2}{\lambda^2}\right\} \tag{22}$$

where the expression in the bracket is a quadratic in $(l+\frac{3}{2})^2$ which becomes negative as l increases. States of higher l can always be constructed by means of

$$\begin{aligned} N_\sigma|E, l, m\rangle &= \langle E, l+2, m+\sigma|N_\sigma|E, l, m\rangle|E, l+2, m+\sigma\rangle \\ &\quad +\langle E, l, m+\sigma|N_\sigma|E, l, m\rangle|E, l, m+\sigma\rangle \\ &\quad +\langle E, l-2, m+\sigma|N_\sigma|E, l, m\rangle|E, l-2, m+\sigma\rangle \end{aligned} \tag{23}$$

unless there exists an integer n such that

$$(E, n\|N\|E, n+2)=0. \tag{24}$$

Combining (24) with (21) gives

$$E_n = k^{1/2}(n+\tfrac{3}{2})+\tfrac{1}{2}\lambda(n^2+3n+\tfrac{3}{2}) \tag{25}$$

where

$$k=\omega^2+\tfrac{1}{4}\lambda^2 \tag{26}$$

and the energy eigenvalues for the oscillator potential are found.† The same reasoning as before shows that an eigenvalue exists for every non-negative n.

Having found the eigenvalues for both potentials it is useful to evaluate the matrix elements in terms of the quantum numbers. If the definition

$$(n, l+1\|R\|n, l)=(l+1)^{1/2}f_{n,l} \tag{27}$$

is made then

$$|f_{n,l}|^2=(n-l)(n+l+2)\{[\mu^2/(n+1)^2]+\lambda(l+1)^2\}. \tag{28}$$

As well, the definition

$$(n, l\|N\|n, l+2)=\frac{[(2l+4)(2l+3)(2l+2)]^{1/2}}{2\cdot 2^{1/2}\omega}C_{n,l} \tag{29}$$

gives

$$|C_{n,l}|^2=(n-l)(n+l+3)[k^{1/2}+\tfrac{1}{2}\lambda(n-l)][k^{1/2}+\tfrac{1}{2}\lambda(n+l+3)]/(l+\tfrac{3}{2})^2. \tag{30}$$

3. Eigenfunction recurrence relations for $N=3$

In the coordinate representation the momentum p_i must be replaced by an hermitian differential operator. The eigenstates become eigenfunctions on the surface of the sphere and they can be chosen to satisfy the invariant normalisation

$$\int \mathrm{d}\boldsymbol{x}\, g^{1/2}\psi^*(\boldsymbol{x})\psi(\boldsymbol{x})=1 \tag{31}$$

† Lakshmanan and Eswaran (1975) have found this result by solving the Schrödinger equation. Their Hamiltonian differs from that of Higgs (1979) by the constant term $\frac{3}{4}\lambda$.

where $0 < r < \infty$ and

$$g(r) = (1+\lambda r^2)^{-4} \tag{32}$$

is the determinant of the metric of the surface. With respect to this normalisation the hermitian momentum operator is

$$p_i = -\mathrm{i} g^{-1/4} \partial_i g^{1/4}. \tag{33}$$

This can be substituted into the constants R_i and N_{ij} to produce differential operators for use in equations (15) and (23).

Using spherical polar co-ordinates the wave functions can be separated into a radial part and an angular part because of the SO(3) symmetry each Hamiltonian possesses. Introducing the radial variable χ (Higgs, 1979), where

$$\tan \chi = \lambda^{1/2} r, \tag{34}$$

this separation is

$$|n, l, m\rangle = X_{n,l}(\chi) Y_{l,m}(\theta, \phi), \tag{35}$$

which can be substituted into either (15) or (23). The orthogonality conditions for the spherical harmonics and some associated properties (e.g. Edmonds 1957) can be used to eliminate the angular dependence from these equations leaving only recurrence relations involving the radial wave functions. Here it is necessary to choose the phases of the reduced matrix elements in a consistent manner as they have not been specified.

After this is done the recurrence relations for the oscillator potential become

$$\begin{aligned}[][(2l+3)\cot\chi\partial_\chi - (2l+3)l\cot^2\chi - l(l+2) + 2E_n/\lambda]X_{n,l}(\chi)\\ = -(2l+3)(|C_{n,l}|/\lambda)X_{n,l+2}(\chi)\end{aligned} \tag{36}$$

and

$$\begin{aligned}\{-(2l+3)\cot\chi\partial_\chi - (2l+3)(l+3)\cot^2\chi - (l+1)(l+3) + 2E_n/\lambda\}\chi_{n,l+2}(\chi)\\ = -(2l+3)(|C_{n,l}|/\lambda)X_{n,l}(\chi)\end{aligned} \tag{37}$$

while for the Coulomb potential the recurrence relations are

$$\{\partial_\chi - l\cot\chi + \mu/[\lambda^{1/2}(l+1)]\}\bar{X}_{n,l}(\chi) = -|f_{n,l}|/[\lambda^{1/2}(l+1)]\bar{X}_{n,l+1}(\chi) \tag{38}$$

and

$$\{-\partial_\chi - (l+2)\cot\chi + \mu/[\lambda^{1/2}(l+1)]\}\bar{X}_{n,l+1}(\chi) = -|f_{n,l}|/[\lambda^{1/2}(l+1)]\bar{X}_{n,l}(\chi). \tag{39}$$

It will be shown in § 5 how these relations may be used to solve fully for the wave functions.

4. Generalisation to N dimensions

The eigenvalue problems have been solved for $N = 2$ (Higgs 1979) and for $N = 3$ earlier in this paper. It should be apparent that considerable difficulty will arise in extending the methods to higher dimensions. However, there is an alternative method based on the Schrödinger equation. The free-particle equation can be written down by introducing the co-ordinate representation into the quantised Hamiltonian of Higgs (1979). If we use hyperspherical polar co-ordinates (e.g. Erdélyi *et al* 1953*b*, p 233) then the equation is separable into a radial equation and an angular equation, the solution of which is a hyperspherical harmonic. The introduction of the two radial potentials does not alter this separability.

This lengthy calculation finally gives the radial Schrödinger equation for the Coulomb potential

$$\{\partial_\chi^2+(N-1)\cot\chi\,\partial_\chi-l(l+N-2)\csc^2\chi+2\hat{\alpha}\cot\chi+2E_n/\lambda\}\bar{X}_{n,l}(\chi)=0 \tag{40}$$

where

$$\hat{\alpha}=\mu/\lambda^{1/2} \tag{41}$$

while for the oscillator potential it is

$$\{\partial_\chi^2+(N-1)\cot\chi\,\partial_\chi-l(l+N-2)\csc^2\chi-(\omega^2/\lambda^2)\tan^2\chi+2E_n/\lambda\}X_{n,l}(\chi)=0. \tag{42}$$

The first equation falls within a class of equations considered by Schrödinger (1940) and later by Infeld and Hull (1951). The method of solution used here is the same although the details are different.

If the differential operators O_+ and O_- are defined by

$$\mathrm{O}_+=\partial_\chi-l\cot\chi+\hat{\alpha}/[l+\tfrac{1}{2}(N-1)] \tag{43}$$

and

$$\mathrm{O}_-=-\partial_\chi-(l+N-1)\cot\chi+\hat{\alpha}/[l+\tfrac{1}{2}(N-1)] \tag{44}$$

then equation (40) can be written in two ways:

$$\mathrm{O}_+\mathrm{O}_-\bar{X}_{n,l+1}-\{\hat{\alpha}^2/[l+\tfrac{1}{2}(N-1)]^2-l(l+N-1)+2E_n/\lambda\}\bar{X}_{n,l+1}=0 \tag{45}$$

and

$$\mathrm{O}_-\mathrm{O}_+\bar{X}_{n,l}-\{\hat{\alpha}^2/[l+\tfrac{1}{2}(N-1)]^2-l(l+N-1)+2E_n/\lambda\}\bar{X}_{n,l}=0. \tag{46}$$

It is clear that $\mathrm{O}_-\bar{X}_{n,l+1}$ is a solution of the second equation while $\mathrm{O}_+\bar{X}_{n,l}$ is a solution of the first so that the operators raise and lower the values of l.

If the eigenfunctions are assumed to be normalised according to the invariant normalisation (31), where

$$g(r)=(1+\lambda r^2)^{-N-1} \tag{47}$$

is the generalisation of (32) and the hyperspherical harmonics are normalised to unity then the radial wave functions must satisfy

$$\lambda^{-N/2}\int_0^{\pi/2}\mathrm{d}\chi(\sin\chi)^{N-1}\bar{X}^*_{n',l}\bar{X}_{n,l}=\delta_{n',n}. \tag{48}$$

With respect to this normalisation the above operators are adjoint, therefore the constant term in (45) and (46) must be non-negative. By the same argument as was used in § 2 there must exist an integer n such that this term is zero when $l=n$. Therefore

$$E_n=-\frac{1}{2}\frac{\mu^2}{(n+\frac{1}{2}(N-1))^2}+\frac{1}{2}\lambda n(n+N-1) \tag{49}$$

for $n=0, 1, 2, \ldots$. This solution corresponds to the previous solutions for $N=2$ and $N=3$.

The action of the raising and lowering operators can now be written as

$$\mathrm{O}_+\bar{X}_{n,l} = -|f_{n,l}|/\{\lambda^{1/2}[l+\tfrac{1}{2}(N-1)]\}\bar{X}_{n,l+1} \tag{50}$$

and

$$\mathrm{O}_-\bar{X}_{n,l+1} = -|f_{n,l}|/\{\lambda^{1/2}[l+\tfrac{1}{2}(N-1)]\}\bar{X}_{n,l} \tag{51}$$

where

$$|f_{n,l}|^2 = \lambda(n-l)(n+l+N-1)\{\hat{\alpha}^2/[n+\tfrac{1}{2}(N-1)]^2+[l+\tfrac{1}{2}(N-1)]^2\} \tag{52}$$

which are the generalisations of the equations obtained in § 3.

The oscillator equation (42) is not one of those considered by Schrödinger (1940). However we can still define the operators

$$\hat{\mathrm{O}}_+ = (2l+N)\cot\chi\,\partial_\chi - (2l+N)l\cot^2\chi - l(l+N-1)+2E_n/\lambda \tag{53}$$

and

$$\hat{\mathrm{O}}_- = -(2l+N)\cot\chi\,\partial_\chi - (2l+N)(l+N)\cot^2\chi - (l+1)(l+N)+2E_n/\lambda \tag{54}$$

such that equation (42) can be written as

$$\hat{\mathrm{O}}_+\hat{\mathrm{O}}_-X_{n,l+2} - \{[2E_n/\lambda-(l+1)(l+N)][2E_n/\lambda - l(l+N-1)] - (2l+N)^2\omega^2/\lambda^2\}X_{n,l+2} = 0 \tag{55}$$

and as

$$\hat{\mathrm{O}}_-\hat{\mathrm{O}}_+X_{n,l} - \{[2E_n/\lambda-(l+1)(l+N)][2E_n/\lambda - l(l+N-1)] - (2l+N)^2\omega^2/\lambda^2\}X_{n,l} = 0. \tag{56}$$

Once again we have raising and lowering operators. The $X_{n,l}$ will satisfy (48) but they will also satisfy the condition that

$$\int_0^{\pi/2} \mathrm{d}\chi(\sin\chi)^{N-1}\csc^2\chi X^*_{n,l'}X_{n,l} = 0 \tag{57}$$

if $l' \neq l$. This can be shown using equations (42) and (48) and it in turn can be used to demonstrate that the raising and lowering operators are adjoint. Therefore the constant term in (55) and (56) is non-negative. By the usual arguments the energy eigenvalues are

$$E_n = (n+\tfrac{1}{2}N)k^{1/2} + \tfrac{1}{2}\lambda(n^2+Nn+\tfrac{1}{2}N) \tag{58}$$

which agrees with the results found for $N=2$ and $N=3$.

Finally the action of the raising and lowering operators is

$$\hat{\mathrm{O}}_+X_{n,l} = -[(2l+N)|C_{n,l}|/\lambda]X_{n,l+2} \tag{59}$$

and

$$\hat{\mathrm{O}}_-X_{n,l+2} = -[(2l+N)|C_{n,l}|/\lambda]X_{n,l} \tag{60}$$

where

$$(l+\tfrac{1}{2}N)^2|C_{n,l}|^2 = \lambda^2(n-l)(n+l+N)[k^{1/2}/\lambda+\tfrac{1}{2}(n-l)][k^{1/2}/\lambda+\tfrac{1}{2}(n+l+N)] \tag{61}$$

which corresponds to that obtained in § 3 for $N=3$.

5. Calculation of the eigenfunctions

We can now make use of the raising and lowering operators to compute the eigenfunctions by noting that for both sets of eigenfunctions the action of the raising operator on the state $l=n$ produces a first order differential equation which can be solved easily. From this state the other states can be generated by successive applications of the lowering operator. This technique is commonly used to generate spherical harmonics (Edmonds 1957). However, the difficulty here arises in writing the eigenfunctions in a closed form. These will now be obtained.

5.1. The oscillator potential

The recurrence relations (59) and (60) can be greatly simplified by the definitions

$$X_{n,l}=A_{n,l}\,(\sin\chi)^{-(l+N-2)}\,(\cos\chi)^{k^{1/2}/\lambda+\frac{1}{2}}Z_{n,l} \tag{62}$$

and

$$A_{n,l-2}=\frac{\lambda A_{n,l}}{(2l+N-4)|C_{n,l-2}|}. \tag{63}$$

With these (60) becomes

$$\begin{aligned} Z_{n,l-2}(\chi)=\csc^2\chi\{&(2l+N-4)\cot\chi\,\partial_\chi \\ &-(n+l+N-2)(2k^{1/2}/\lambda+n-l+2)\}Z_{n,l}(\chi) \end{aligned} \tag{64}$$

while for $l=n$ (59) becomes

$$\{\partial_\chi-(2n+N-2)\cot\chi\}Z_{n,n}(\chi)=0 \tag{65}$$

which has the solution

$$Z_{n,n}(\chi)=(\sin\chi)^{(2n+N-2)}. \tag{66}$$

If $X_{n,n}$ is normalised according to (48) then all the other eigenfunctions will be correctly normalised by means of (63). It remains therefore to compute the other $Z_{n,l}$ from $Z_{n,n}$. It can be shown by induction that

$$Z_{n,n-2D}=\sum_{r=0}^{D}(-1)^r B_{D,r}^{n,N}(\sin\chi)^{2n+N-4D+2r-2}(\cos\chi)^{2D-2r} \tag{67}$$

where

$$B_{D,r}^{n,N}=\frac{2^{2r-2D}D!\Gamma(k^{1/2}/\lambda+D+1)}{(D-r)!r!\Gamma(k^{1/2}/\lambda+D+1-r)}\frac{\Gamma(2n+N-1)\Gamma(\frac{1}{2}(2n+N-4D+2r-1))}{\Gamma(\frac{1}{2}(2n+N-1))\Gamma(2n+N-4D+2r-1)}. \tag{68}$$

At this point we can change to the variable

$$Z=\sin^2\chi \tag{69}$$

and prove the following result

$$\begin{aligned} &(\mathrm{d}/\mathrm{d}Z)^D[Z^{\frac{1}{2}(2n+N-2D-2)}(1-Z)^{k^{1/2}/\lambda+D}] \\ &\quad=\frac{\Gamma(2n+N-2D-1)\Gamma[\frac{1}{2}(2n+N-1)]}{\Gamma[\frac{1}{2}(2n+N-2D-1)]\Gamma(2N+N-1)}(1-Z)^{k^{1/2}/\lambda}Z_{n,n-2D}(Z). \end{aligned} \tag{70}$$

When the normalisation constant is evaluated the result for the radial eigenfunction is

$$X_{n,l}(Z) = \left[\frac{2\lambda^{N/2}(k^{1/2}/\lambda + n + \frac{1}{2}N)\Gamma[k^{1/2}/\lambda + \frac{1}{2}(n+l+N)]}{[\frac{1}{2}(n-l)]!\Gamma[\frac{1}{2}(n+l+N)]\Gamma[k^{1/2}/\lambda + \frac{1}{2}(n-l)+1]}\right]^{1/2}$$
$$\times Z^{-\frac{1}{2}(l+N-2)}(1-Z)^{-k^{1/2}/2\lambda+\frac{1}{4}}$$
$$\times (\mathrm{d}/\mathrm{d}Z)^{\frac{1}{2}(n-l)}[Z^{\frac{1}{2}(n+l+N-2)}(1-Z)^{k^{1/2}/\lambda+\frac{1}{2}(n-l)}] \tag{71}$$

which can in turn be written as

$$\text{const} \times \frac{\Gamma[\frac{1}{2}(n+l+N)]}{\Gamma[\frac{1}{2}(2l+N)]} Z^{\frac{1}{2}l}(1-Z)^{k^{1/2}/2\lambda+\frac{1}{4}}$$
$$\times F(k^{1/2}/\lambda + \tfrac{1}{2}(n+l+N), -\tfrac{1}{2}(n-l); l+\tfrac{1}{2}N; Z) \tag{72}$$

using the hypergeometric function relations of Erdélyi *et al* (1953a, p 101).

5.2. The Coulomb potential

It will be shown here that the recurrence relations (50) and (51) generate radial eigenfunctions of the form

$$\bar{X}_{n,l}(\chi) = A_{n,l}\frac{(2N+N-2)!}{(n+l+N-2)!}(\sin\chi)^{-(n+N-1)}\exp(\tfrac{1}{2}\hat{\beta}\chi)[(\sin\chi)^2(\mathrm{d}/\mathrm{d}\chi)]^{(n-l)}$$
$$\times[(\sin\chi)^{2l+N-1}\exp(-\hat{\beta}\chi)] \tag{73}$$

where

$$A_{n,l-1} = \frac{\lambda^{1/2}}{2|f_{n,l-1}|}A_{n,l} \tag{74}$$

and

$$\hat{\alpha} = \tfrac{1}{2}(n+\tfrac{1}{2}(N-1))\hat{\beta}. \tag{75}$$

This certainly satisfies the equation

$$\hat{\mathrm{O}}_{+}\bar{X}_{n,n} = 0 \tag{76}$$

so there remains to be proven that it satisfies (51).

The first step in this process is to prove that

$$[(\sin\chi)^2(\mathrm{d}/\mathrm{d}\chi)]^{2D}[(\sin\chi)^{2n-4D+N-1}\exp(-\hat{\beta}\chi)]$$
$$= \frac{(2n-2D+N-2)!}{(2n+N-2)!}(\sin\chi)^{2n-2D+N-1}(\cos\chi)^{2D}\exp(-\hat{\beta}\chi)$$
$$\times\left(\sum_{S=0}^{D}\hat{\beta}^{2S}\sum_{r=0}^{D-S}(-1)^r G_{r,S}^{D,n}(\tan\chi)^{2r+2S}\right.$$
$$\left.-\sum_{S=0}^{D}\hat{\beta}^{2S+1}\sum_{r=0}^{D-S}(-1)^r H_{r,S}^{D,n}(\tan\chi)^{2r+2S+1}\right) \tag{77}$$

and

$$[(\sin\chi)^2\,(\mathrm{d}/\mathrm{d}\chi)]^{2D+1}[(\sin\chi)^{2n-4D+N-3}\exp(-\hat{\beta}\chi)]$$

$$=\frac{(2n+N-2D-3)!}{(2n+N-2)!}(\sin\chi)^{2n-2D+N-2}(\cos\chi)^{2D+1}\exp(-\hat{\beta}\chi)$$

$$\times\left(\sum_{S=0}^{D}\hat{\beta}^{2S}\sum_{r=0}^{D-S}(-1)^r E_{r,S}^{D,n}(\tan\chi)^{2r+2S}\right.$$

$$\left.-\sum_{S=0}^{D}\hat{\beta}^{2S+1}\sum_{r=0}^{D-S}(-1)^r F_{r,S}^{D,n}(\tan\chi)^{2r+2S+1}\right) \tag{78}$$

where

$$G_{r,S}^{D,n}=\frac{1}{(r+S)!}\frac{(2D)!}{(2D-2r-2S)!}g_{r,S}^{D,n}\frac{(2n+N-2)!}{(2n-4D+N+2r+2S-2)!}, \tag{79}$$

$$H_{r,S}^{D,n}=\frac{1}{(r+S)!}\frac{(2D)!}{(2D-2r-2S-1)!}h_{r,S}^{D,n}\frac{(2n+N-2)!}{(2n-4D+N+2r+2S-1)!}, \tag{80}$$

$$E_{r,S}^{D,n}=\frac{1}{(r+S)!}\frac{(2D+1)!}{(2D-2r-2S+1)!}g_{r,S}^{D,n-1}\frac{(2n+N-2)!}{(2n-4D+N+2r+2S-4)!}, \tag{81}$$

$$F_{r,s}^{D,n}=\frac{1}{(r+S)!}\frac{(2D+1)!}{(2D-2r-2S)!}h_{r,s}^{D,n-1}\frac{(2n+N-2)!}{(2n-4D+N+2r+2S-3)!}. \tag{82}$$

and where

$$g_{r,0}^{D,n}=\frac{\Gamma[\frac{1}{2}(2n-4D+N+2r-1)]}{\Gamma[\frac{1}{2}(2n-4D+N-1)]}, \tag{83}$$

$$g_{r,S}^{D,n}=\frac{1}{2}\sum_{t=0}^{r}\frac{\Gamma(\frac{1}{2}(2n-4D+N+2r+2S-1))}{\Gamma(\frac{1}{2}(2n-4D+N+2t+2S-1))}h_{t,S-1}^{D,n}, \qquad S\neq 0 \tag{84}$$

and

$$h_{r,S}^{D,n}=\sum_{t=0}^{r}\frac{(r+S)!}{(t+S)!}\frac{(r+S)!}{(t+S)!}\frac{(2t+2S)!}{(2r+2S+1)!}$$

$$\times\frac{\Gamma(2n-4D+N+2r+2S-1)}{\Gamma(2n-4D+N+2t+2S-1)}\frac{\Gamma[\frac{1}{2}(2n-4D+N+2t+2S-1)]}{\Gamma[\frac{1}{2}(2n-4D+N+2r+2S-1)]}g_{t,S}^{D,n}. \tag{85}$$

This can be done by noting that

$$[(\sin\chi)^2(\mathrm{d}/\mathrm{d}\chi)]^{2D+1}\{(\sin\chi)^{2n+N-4D-1}\exp(-\hat{\beta}\chi)\}$$

and

$$[(\sin\chi)^2(\mathrm{d}/\mathrm{d}\chi)]^{2D+2}\{(\sin\chi)^{2n+N-4D-1}\exp(-\hat{\beta}\chi)\}$$

can each be written in two different ways. This will provide relations involving the coefficients which are satisfied by those given here.

The second step is to substitute (77) and (78) into (73) and prove that they satisfy the recurrence relation. Once again this will produce a number of relations involving the coefficients given in equations (79) to (82) which can be reduced to two equations involving the subsidiary coefficients given in equations (83) to (85), these being

$$g_{r,S}^{D,n-1} = g_{r,S}^{D,n} - (r+S)g_{r-1,S}^{D,n} \tag{86}$$

and

$$h_{r,S}^{D,n-1} = h_{r,S}^{D,n} - (r+S)h_{r-1,S}^{D,n}. \tag{87}$$

The equations can be proven by induction on S.

The constants $A_{n,l}$ can be evaluated from $A_{n,n}$ using (74) while $A_{n,n}$ can be evaluated by normalising $\bar{X}_{n,n}$ according to (48). Thus the proof that the radial eigenfunctions for the Coulomb potential have the form (73) is complete.

6. The dynamical symmetry groups of the quantum mechanical system

6.1. *The Coulomb problem*

For a given energy eigenvalue E_n, the possible eigenvalues of $\frac{1}{2}L_{ij}L_{ij}$ are $l(l+N-2)$ where $l = 0, 1, \ldots, n$. Therefore, as for the n-th discrete eigenvalue when $\lambda = 0$, the corresponding energy eigenspace carries a direct sum of those irreducible representations of SO(N) labelled $(l, 0^{\nu-1})$ (Boerner 1963) for $l = 0, 1, \ldots, n$, where $N = 2\nu$ if N is even and $N = 2\nu+1$ if N is odd.† This is precisely the SO(N) content of the irreducible representation of SO($N+1$) labelled $(n, 0^{\sigma-1})$, where $N = 2\sigma - 1$ if N is odd and $N = 2\sigma$ if N is even. On this representation, the second order Casimir operator C of SO($N+1$) takes the value $n(n+N-1)$ (Perelemov and Popov, 1966b). It follows that there exist hermitian operators M_i $i = 1, 2, \ldots, N$ which, together with the L_{ij}, generate in each energy eigenspace the corresponding representation of SO($N+1$). These operators therefore satisfy

$$[L_{ij}, M_k] = \mathrm{i}(\delta_{ik}M_j - \delta_{jk}M_i), \tag{88}$$

$$[M_i, M_j] = \mathrm{i}L_{ij}, \tag{89}$$

$$[H, M_i] = 0 \tag{90}$$

and, as can be seen by comparing the eigenvalues of H in equation (49) and of C above,

$$H = \tfrac{1}{2}\lambda C - \tfrac{1}{2}\mu^2[\epsilon + \tfrac{1}{4}(N-1)^2]^{-1} \tag{91}$$

(c.f. Higgs 1979, equation (26a)), where

$$C = M_iM_i + \tfrac{1}{2}L_{ij}L_{ij}. \tag{92}$$

It can be seen from comparing (89) and (8) that it is not a straightforward matter to construct M_i from R_i when $\lambda \neq 0$, and indeed we have not been able to identify M_i explicitly except for $N = 2$ (Higgs 1979).

6.2. *The Oscillator problem*

Similar considerations apply here except that the possible eigenvalues of $\frac{1}{2}L_{ij}L_{ij}$ are $l(l+N-2)$ where $l = n, n-2, \ldots, 1$ or 0. Therefore the energy eigenspace carries a

† The case $N = 2$ is special. There the representations of SO(2) which occur are those labelled (l) for $l = n, n-1, \ldots, -n$ forming a basis for the representation (n) of SO(3) (Higgs, 1979).

direct sum of the $(l, 0^{\nu-1})$ representations of SO(N) where $l = n, n-2, \ldots, 1$ or 0.† This is the SO(N) content of the irreducible symmetric representation of SU(N) labelled $(n, 0^{N-2})$ (Hamermesh 1962), on which the second order Casmir operator C of SU(N) takes the value $n(n+N)$ (Perelemov and Popov, 1966*a*). Therefore there exist hermitian operators $\hat{N}_{ij}$, symmetric in the indices and traceless, which, together with the L_{ij}, generate these representations of SU(N). These operators satisfy

$$[L_{ij}, \hat{N}_{kl}] = \mathrm{i}(-\delta_{jk}\hat{N}_{il} - \delta_{jl}\hat{N}_{ik} + \delta_{ik}\hat{N}_{jl} + \delta_{il}\hat{N}_{jk}), \tag{93}$$

$$[\hat{N}_{ij}, \hat{N}_{kl}] = \mathrm{i}(\delta_{jk}L_{il} + \delta_{jl}L_{ik} + \delta_{ik}L_{jl} + \delta_{il}L_{jk}), \tag{94}$$

$$[H, \hat{N}_{ij}] = 0, \tag{95}$$

and

$$H = \tfrac{1}{2}\lambda(C + \tfrac{1}{2}N) + \{(\omega^2 + \tfrac{1}{4}\lambda^2)(C + \tfrac{1}{4}N^2)\}^{1/2} \tag{96}$$

(cf Higgs, 1979, equation (47a)), where

$$4(1 - N^{-1})C = N_{ij}N_{ij} + L_{ij}L_{ij}. \tag{97}$$

Once again we have not been able to construct these $\hat{N}_{ij}$ from the N_{ij} introduced by Higgs (1979).

7. Discussion

Higgs (1979) has demonstrated that, classically, the constants of the motion associated with the Coulomb and oscillator potentials on a sphere can be written in a form exhibiting explicitly the dynamical symmetry groups. He has also shown that it can be done quantum-mechanically in two dimensions. It has not been done more generally because of the problems associated with the ordering of non-commuting operators. However it has been shown in this paper that these problems can be avoided by considering the matrix elements of the operators. Further, it has been shown that there is an alternative method due to Schrödinger (1940) which further reduces the complexity of the problem. Thus it is clear that even though the symmetries can be demonstrated only indirectly, they have provided a powerful technique for solving the two problems.

Acknowledgments

The author gratefully acknowledges the assistance and guidance provided by Dr P W Higgs and the staff of the Department of Physics at the University of Edinburgh, the financial support of the Commonwealth Scholarship and Fellowship Plan during his stay there, and, most recently, the many valuable discussions with Dr A J Bracken of the Department of Mathematics at the University of Queensland.

† Once again the case $N = 2$ is a special one. The representations of SO(2) that occur are those labelled (l), $l = n, n-2, \ldots, -n$, forming a basis for the representation (n) of SU(2).

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